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ON PRIMARY FACTORIZATIONS*

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We relate ideals in commutative rings which are products of primary ideals to ideals with primary decompositions. Invertible primary ideals are shown to have special properties. Sufficient conditions are given for a primary product ideal to have a unique product representation. A domain is weakly factorial if every non-unit is a product of primary elements. If R is weakly factorial, $\mathrm{Pic}(R)=0$. A Noetherian weakly factorial domain R is factorial precisely when R is integrally closed. $R[X]$ is weakly factorial if and only if R is a weakly factorial GCD domain. Properties of weakly factorial GCD domains are discussed.

1. Introduction

Throughout this paper all rings will be assumed to be commutative with identity. We first discuss ideals that are a product of primary ideals and relate primary products to primary decompositions. An ideal with a primary decomposition need not be a product of primary ideals and an ideal that is a product of primary ideals need not have a primary decomposition. However, we show that if an ideal generated by an R-sequence is a product of primary ideals, then it has a primary decomposition. We also show that if Q_1 and Q_2 are P-primary and Q_1 is locally principal, then Q_1Q_2 is P-primary. Moreover we show that distinct prime ideals P_1 and P_2 with respective invertible primary ideals Q_1 and Q_2 are incomparable. We conclude Section two with a discussion of uniqueness of primary factorizations. If I is an ideal in R with $I=Q_1Q_2\cdots Q_n$ where Q_i is P_i-primary, then this product is said to be a *reduced primary product representation* if $P_i\neq P_j$, $i\neq j$, and $I\neq Q_1\cdots Q_{j-1}Q_{j+1}\cdots Q_n$. We show that if I is a reduced product of primary ideals which are either invertible or have maximal radicals, then this factorization is unique. Thus if D is a Q-domain,

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i.e., D is an integral domain in which every ideal is a product of primary ideals, then every ideal is the unique product of primary ideals.

In the third section we continue our study of primary factorization by examining rings in which every regular nonunit is a product of primary elements. (An element x is primary if (x) is a primary ideal.) We call integral domains with this property *weakly factorial*. Invertible ideals generated by regular elements in a ring R in which every regular nonunit is a product of primary elements are shown to be principal and hence if R is either Noetherian or an integral domain, $\mathrm{Pic}(R)=0$. We show that for an integral domain R, every proper principal ideal of R has a primary decomposition involving only height one prime ideals if and only if $R=\bigcap_{\mathrm{ht}(P)=1} R_P$ where the intersection is locally finite. Moreover for a weakly factorial domain R, to be factorial is equivalent to being a Krull domain which is in turn equivalent to having the property that for each height one prime ideal P, R_P is a DVR. An integral domain R is a weakly factorial GCD domain if and only if $R[X]$ is weakly factorial. Also for a weakly factorial domain R to be a GCD domain it is necessary and sufficient that R_P be a valuation domain for each height one prime ideal P of R.

Weakly factorial GCD domains are completely characterized as generalized Krull domains. (R is a generalized Krull domain if $R=\bigcap_\lambda V_\lambda$ where each V_λ is an essential rank one valuation overring of R and the intersection is locally finite.)

A proper ideal of R is an ideal I such that $I \subsetneq R$. For other terminology we refer the reader to [5].

2. Products of primary ideals

Every ideal in a Noetherian ring has a primary decomposition, but such an ideal need not be a product of primary ideals. For example, let $R=K[X,Y,Z]$ where K is a field. The ideal $(X,Y^2,YZ)=(X,Y,Z)(X,Y)+(X)$ has a reduced primary decomposition $(X,Y)\cap(X,Y^2,Z)$, but it is *not* a product of primary ideals. (Observe that (X,Y^2,YZ) is not a multiple of (X,Y) and apply [1, Lemma 3].)

A ring R is defined to be a *Q-ring* if every proper ideal is a product of primary ideals. In a previous paper we have shown that R is a Q-ring if and only if R is Laskerian (every proper ideal of R has a primary decomposition) and every nonmaximal prime ideal of R is finitely generated and locally principal [3, Theorem 13]. Thus if every proper ideal of a ring is a product of primary ideals, then every proper ideal has a primary decomposition. However, an ideal can be a product of primary ideals without having a primary decomposition. For example, let $\mathbb{Z}$ be the ring of integers and $M=\bigoplus \mathbb{Z}/p\mathbb{Z}$ where the sum runs over all nonzero primes p of $\mathbb{Z}$. Let $R=Z\oplus M$ be the idealization of $\mathbb{Z}$ and M. Then $0\oplus M$ is a prime ideal of R so $0=(0\oplus M)^2$ is a product of prime (and hence primary) ideals, but 0 does *not* have a primary decomposition since $\mathbb{Z}(R)=\bigcup((p)\oplus M)$ is not a finite union of prime ideals. However, we will show that if an ideal generated by an R-sequence is a product of primary ideals, then it does have a primary decomposition.

With a little more work we can construct an example of an integral domain R with a prime ideal P such that P^2 does not have a primary decomposition. Let $\mathscr{P}=\{p_1, p_2, \ldots\}$ be the set of positive primes. Let $R=\mathbb{Z}[\{X_{p_i}, \sqrt{p_i}X_{p_i}^{1/2} \mid p_i \in \mathscr{P}\}]$ and let $P=(\{X_{p_i}, \sqrt{p_i}X_{p_i}^{1/2} \mid p_i \in \mathscr{P}\})$. Since $R/P \simeq \mathbb{Z}$, P is a prime ideal of R. Now $P^2=(\{X_{p_i}X_{p_j}, p_iX_{p_i}, \sqrt{p_ip_j}X_{p_i}^{1/2}X_{p_j}^{1/2}\})$. Let $M_i=(P, p_i)$ so $R/M_i \simeq \mathbb{Z}/p_i\mathbb{Z}$ and hence M_i is a maximal ideal of R. Now $M_i \subseteq \mathbb{Z}(R/P^2)$ since $p_iX_{p_i} \in P^2$. Hence $\mathbb{Z}(R/P^2)$ is *not* a finite union of prime ideals, so P^2 does *not* have a primary decomposition. It is interesting to note that P^2 is however an infinite intersection of primary ideals. Indeed, it can be shown that $P^2=P^{(2)} \cap (\bigcap_i Q_i)$ where $Q_i=(\{X_{p_j}\}_{j \neq i}, X_{p_i}^2, \sqrt{p_j}X_{p_j}^{1/2}, p_i)$ is M_i-primary.

Now in general the product of two P-primary ideals need not be P-primary. However, in the special case where the P-primary ideals are regular principal ideals, their product is again P-primary. This result will follow from some more general considerations.

Theorem 1. *Let P be a prime ideal and Q_1 and Q_2 be P-primary. Suppose that for each maximal ideal M of R, Q_{1M} is a regular principal ideal. Then Q_1Q_2 is P-primary.*

Proof. First suppose that Q_1 is a regular principal ideal, say $Q_1=(q_1)$. Then $\sqrt{(q_1)Q_2}=P$. Let $xy \in (q_1)Q_2$, say $xy=q_1q_2$, $q_2 \in Q_2$, with $y \notin P$. We need that $x \in (q_1)Q_2$. Now $xy \in (q_1)Q_2 \subseteq (q_1)$ and $y \notin P$ implies that $x \in (q_1)$, say $x=tq_1$. Then $tq_1y=xy=q_1q_2$ implies $ty=q_2 \in Q_2$. So $y \notin P$ implies $t \in Q_2$. Hence $x=tq_1 \in (q_1)Q_2$.

We next do the general case. First $\sqrt{Q_1Q_2}=P$. It suffices to show that for each maximal ideal $M \supseteq P$, $(Q_1Q_2)_M$ is P_M-primary. For if $xy \in Q_1Q_2$ and $y \notin P$, then $xy/1 \in (Q_1Q_2)_M$ and $y/1 \notin P_M$ so $x/1 \in (Q_1Q_2)_M$. Thus $x \in Q_1Q_2$. But Q_{1M} is a regular principal primary ideal, so by the preceding paragraph $Q_{1M}Q_{2M}=(Q_1Q_2)_M$ is P_M-primary. $\square$

Corollary 2. *Let Q_1 and Q_2 be P-primary. If Q_1 is invertible, then Q_1Q_2 is P-primary.* $\square$

An ideal I is a *multiplication ideal* if for each ideal $J \subseteq I$ there is an ideal K with $J=IK$. A principal ideal or invertible ideal is easily seen to be a multiplication ideal.

Theorem 3. *Let $P_1, \ldots, P_n$ be distinct prime ideals and let Q_i be P_i-primary. If each Q_i is a multiplication ideal, then $Q_1 \cap \cdots \cap Q_n = Q_1 \cdots Q_n$.*

Proof. The proof of Theorem 3 is essentially the same as the proof of [5, Theorem 43.15]. Rearrange $P_1, \ldots, P_n$, if necessary, so that $P_i \not\subseteq P_j$ for $i<j$. Since $Q_1 \supseteq Q_1 \cap Q_2$ and Q_1 is a multiplication ideal, $Q_1 \cap Q_2 = Q_1B$ for some ideal B. But $Q_1B=Q_1 \cap Q_2 \subseteq Q_2$ and $Q_1 \not\subseteq P_2$ implies $B \subseteq Q_2$ so $Q_1 \cap Q_2 = Q_1Q_2$. By induction assume that $Q_1 \cap \cdots \cap Q_k = Q_1 \cdots Q_k$. Since $Q_1 \cdots Q_k$ is a multiplication ideal [2, Corollary, p. 466], $(Q_1 \cdots Q_k) \cap Q_{k+1} = Q_1 \cdots Q_kB$ for some ideal B. But then

$Q_1 \cdots Q_k B \subseteq Q_{k+1}$ and $Q_i \not\subseteq P_{k+1}$ for $i=1,\dots,k$ implies that $B \subseteq Q_{k+1}$ and hence $Q_1 \cap \cdots \cap Q_{k+1} = (Q_1 \cap \cdots \cap Q_k) \cap Q_{k+1} = (Q_1 \cdots Q_k) \cap Q_{k+1} \subseteq Q_1 \cdots Q_k Q_{k+1}$ so $Q_1 \cap \cdots \cap Q_{k+1} = Q_1 \cdots Q_{k+1}$. □

Remark. In Theorem 3, $Q_1 \cap \cdots \cap Q_n$ need not be a reduced primary decomposition for $Q_1 \cdots Q_n$. For example, in $R = \mathbb{Z} \oplus \mathbb{Z}$ where $P_1 = Q_1 = 0 \oplus \mathbb{Z} \subsetneq P_2 = Q_2 = (2) \oplus \mathbb{Z}$, we have $P_1 = P_1 P_2 \subsetneq P_2$, so $P_1 \cap P_2$ is not a reduced primary decomposition for $P_1 P_2$. However, if there are no containment relations between $P_1, \dots, P_n$, then $Q_1 \cap \cdots \cap Q_n$ is the reduced primary decomposition for $Q_1 \cdots Q_n$. For if some $Q_{i_0} \supseteq \bigcap_{i \neq i_0} Q_i$, then $Q_{i_0} \supseteq \prod_{i \neq i_0} Q_i$ which implies that some $Q_i \subseteq P_{i_0}$ and hence $P_i \subseteq P_{i_0}$, a contradiction. Hence each P_i is a minimal prime of $Q_1 \cdots Q_n$, so the P_i-primary component is uniquely determined. An interesting case where there are no containment relations is given in the next theorem.

Theorem 4. *Let P_1 and P_2 be distinct prime ideals and let Q_i be P_i-primary for $i = 1, 2$. Suppose that Q_1 and Q_2 are invertible. Then P_1 and P_2 are incomparable.*

Proof. Suppose that $P_1 \subsetneq P_2$. Then $Q_1 \subseteq P_1 \subsetneq P_2 = \sqrt{Q_2}$, so there is a smallest positive integer n with $Q_1^n \subseteq Q_2$. Now Q_2 is invertible, so $Q_1^n = Q_2 A$ for some ideal A. Hence $Q_2 \not\subseteq P_1$, so $A \subseteq Q_1$, say $A = A_1 Q_1$. But then $Q_1^n = Q_2 A = Q_2 Q_1 A_1$ which implies that $Q_1^{n-1} = Q_2 A_1 \subseteq Q_2$, contradicting the minimality of n. □

Corollary 5. *Let $Q_1, \dots, Q_n$ be invertible P_i-primary ideals where $P_1, \dots, P_n$ are distinct prime ideals. Then $Q_1 \cap \cdots \cap Q_n$ is the reduced primary decomposition for $Q_1 \cdots Q_n$.*

Theorem 6. *Let I be an ideal that is a product of primary ideals. If either I is invertible or I is generated by an R-sequence, then I has a primary decomposition.*

Proof. First suppose that I is invertible. Let $I = Q_1 \cdots Q_n$ where Q_i is P_i-primary. Then each Q_i is a factor of an invertible ideal and hence is itself invertible. If for some $i \neq j$, $P_i = P_j$, then by Corollary 2, $Q_i Q_j$ is $P_i = P_j$-primary and is again invertible. Thus we can assume that $P_1, \dots, P_n$ are distinct primes. Then by Corollary 5, I has a primary decomposition.

Next suppose that $I = (x_1, \dots, x_n)$ where $x_1, \dots, x_n$ is an R-sequence. We have already proven the result for the case $n = 1$. So assume $n > 1$ and that the result is true for $n - 1$. Let $I = Q_1 \cdots Q_t$ where Q_i is primary. Pass to $\bar{R} = R/(x_1)$. Then $\bar{I}$ is generated by an R-sequence of length $n - 1$ and $\bar{I} = \bar{Q}_1 \cdots \bar{Q}_t$ is a product of primary ideals. By induction, $\bar{I}$ has a primary decomposition. Hence I has a primary decomposition. □

Corollary 7. *Let (f) be a regular principal ideal that is a product of primary ideals. Then (f) has a primary decomposition.* □

Of course a regular principal ideal can have a primary decomposition without being a product of primary ideals. For example, let R be a Krull domain with a height one prime ideal P that is *not* invertible. Let $x \in P - P^2$. Now certainly (x) has a primary decomposition. But if $(x) = Q_1 \cdots Q_n$ where each Q_i is primary, then some $Q_i \subseteq P$. Since $x \in P - P^2$, $Q_i = P$. But since $P = Q_i$ is a factor of (x), it must be invertible, a contradiction.

We end this section with a discussion of uniqueness of products of primary ideals. If an ideal I is a proper, finitely generated regular ideal of a commutative ring R such that I is a product of proper prime ideals of R, then this representation as a finite product of prime ideals is unique [5, Theorem 37.14]. In general, this concept of uniqueness does not extend to products of primary ideals, for if M is a maximal ideal, then M^2 is primary but $M^2 = MM$ has two factorizations as a product of primary ideals. Thus we will consider an ideal I to be a unique product of primary ideals if the components associated with each prime ideal are unique. To facilitate this discussion we say that if an ideal $I = Q_1 \cdots Q_n$ where Q_i is P_i-primary, then this product decomposition for I is a *reduced primary product representation* if $P_i \neq P_j$ for $i \neq j$ and $I \neq Q_1 \cdots Q_{j-1} Q_{j+1} \cdots Q_n$ (i.e., none of the Q_i may be deleted).

It is well known that if R is a Noetherian ring in which every (nonzero) proper prime ideal is maximal, then every nonzero ideal is in a unique way a product of primary ideals (i.e., if $I = Q_1 \cap Q_2 \cap \cdots \cap Q_n$, then $I = Q_1 Q_2 \cdots Q_n$ [7, p. 213]). Moreover since the proof of this result uses the Noetherian hypothesis only to guarantee a primary decomposition, the result is readily generalized to the Laskerian case.

In general, however, even for Noetherian domains of Krull dimension two, the primary decomposition for an ideal I is uniquely determined by I if and only if I has no embedded components. We will show that for a Q-domain the factorization of an ideal I into the product of primary components is unique. We begin with a slightly more general result.

Theorem 8. *Let I be an ideal of R. Suppose that I is a reduced product of primary ideals, $I = Q_1 \cdots Q_n$, where Q_i is P_i-primary and either Q_i is invertible or P_i is maximal. Then this factorization of I into primary ideals is unique in the following sense: if $I = Q_1' \cdots Q_{n'}'$ (reduced) where Q_i' is P_i'-primary, then $n = n'$ and $\{Q_1, \dots, Q_n\} = \{Q_1', \dots, Q_n'\}$.*

Proof. Suppose that $Q_1, \dots, Q_s$ are invertible and that $P_{s+1}, \dots, P_n$ are maximal ideals. For $1 \le i \le s$, $Q_{iP_i} = I_{P_i} = Q'_{1P_i} \cdots Q'_{n'P_i}$ since, by Theorem 4, there are no containment relations between the primes $P_1, \dots, P_s$. Since the primes $P_1', \dots, P_n'$ are distinct, we must have $Q_{iP_i} = Q'_{j'P_i}$ for some j. Because Q_i and $Q_{j'}$ are primary, $Q_i = Q_{j'}$. Reordering, if necessary, we have $Q_1' = Q_1, \dots, Q_s' = Q_s$. Thus $(Q_1 \cdots Q_s)(Q_{s+1} \cdots Q_n) = I = (Q_1 \cdots Q_s)(Q_{s+1}' \cdots Q_{n'}')$. Since $Q_1 \cdots Q_s$ is invertible, we have $Q_{s+1} \cdots Q_n = Q_{s+1}' \cdots Q_{n'}'$. Moreover, $P_{s+1}, \dots, P_n$ are all maximal ideals. Thus for $s+1 \le i \le n'$, $P_i' \supseteq Q_i' \supseteq Q_{s+1} \cdots Q_n$, so $P_i' \supseteq P_j$, for some j $(s+1 \le j \le n)$, and thus $P_i' = P_j$ since P_j is maximal. Thus each P_i' $(s+1 \le i \le n')$ is a maximal ideal and $\{P_{s+1}', \dots, P_{n'}'\} \subseteq$

$\{P_{s+1},\ldots,P_n\}$. By reversing the roles of P_i and P'_i, $\{P_{s+1},\ldots,P_n\}\subseteq\{P'_{s+1},\ldots,P'_{n'}\}$ so $n=n'$ and $\{P_{s+1},\ldots,P_n\}=\{P'_{s+1},\ldots,P'_{n'}\}$. Reordering, if necessary, we can assume that $P_i=P'_i$. But then for $s+1\le i\le n$, $Q_{iP_i}=(Q_{s+1}\cdots Q_n)_{P_i}=(Q'_{s+1}\cdots Q'_n)_{P_i}=Q'_{iP_i}$ so $Q_i=Q'_i$. □

Note that invertible primary ideals exist whose prime radicals are not invertible. For if $R=K[[X^2,X^3]]$, K a field, (X^2) is (X^2,X^3)-primary but (X^2,X^3) is not principal and thus not invertible since R is a local domain.

Theorem 9. *If D is a Q-domain, then every proper ideal I of D has a unique reduced primary product representation.*

Proof. If D is a Q-domain, then D is Laskerian and every nonmaximal prime ideal is invertible. Let I be a proper ideal of D, $I=Q_1\cdots Q_n$ where Q_i is P_i-primary; the factorization can be reordered if necessary so that for $i=1,\ldots,m$, P_i is invertible and for $i=m+1,\ldots,n$, P_i is maximal. In either case, if $P_i=P_j$, then Q_iQ_j is $P_i=P_j$-primary and the primary factorization is componentwise unique by Theorem 8. □

Example. Consider the 2-dimensional factorial domain $K[X,Y]$ where K is a field. Since every nonmaximal prime ideal of $K[X,Y]$ is principal, $K[X,Y]$ is a Q-domain. By Theorem 9, every proper ideal of $K[X,Y]$ has a unique reduced primary product representation. It is interesting to note that while the ideal $(X^3,XY)=(X)\cap(X^3,Y+cX^2)$, $c\in K$, has more than one reduced primary decomposition, it has the unique reduced primary product representation $(X^3,XY)=(X)(X^2,Y)$.

3. Weakly factorial domains

An element x of a ring R will be called *primary* if (x) is a primary ideal. In this section we study rings with the property that every regular nonunit is a product of primary elements (or equivalently that every proper regular principal ideal is a product of principal primary ideals). An integral domain R will be called *weakly factorial* if every nonunit is a product of primary elements. Of course a factorial domain is weakly factorial, but since any one-dimensional quasilocal domain is clearly weakly factorial, the converse is false.

The following observation will prove useful. Suppose that (x) is a regular principal ideal that is a product of principal primary ideals, say $(x)=(q_1)\cdots(q_t)$ where (q_i) is P_i-primary. If some $P_i=P_j$ for $i\neq j$, then (q_iq_j) is again $P_i=P_j$-primary. Thus by combining primaries for the same prime, we can assume that the P_i's are distinct primes. Then this factorization into primaries is unique in the sense of Theorem 8 and $(x)=(q_1)\cap\cdots\cap(q_t)$ is *the* reduced primary decomposition for (x).

It is well known that every invertible ideal in a factorial domain is principal. Our next result shows that this is also true for weakly factorial domains.

Theorem 10. *Let R be a ring in which every regular nonunit is a product of primary elements. Suppose that I is an invertible ideal of R that is generated by regular elements. Then I is principal.*

Proof. Let $x \in I$ be regular. Then $x = q_1 \cdots q_m$ where (q_i) is $\mathscr{P}_i$-primary. Let P be a minimal prime ideal of I. Now some $q_i \in P$, so $\mathscr{P}_i = \sqrt{(q_i)} \subseteq P$. Moreover, I_P is a regular principal ideal and is P_P-primary. Also, $(q_i)_P$ is a regular principal ideal that is $\mathscr{P}_{iP}$-primary. By Theorem 4, $\mathscr{P}_{iP} = P_P$ and hence $P = \mathscr{P}_i$. Thus there are only finitely many primes minimal over I, say $P_1, \ldots, P_n$. For each $1 \leq i \leq n$, let $Q_i = I_{P_i} \cap R$, so Q_i is P_i-primary. Since I is generated by regular elements, $I_{P_i} = (r_i)_{P_i}$ for some regular element $r_i \in I$. By hypothesis, r_i is a product of primary elements. In fact, we can write $r_i = q_i' \cdots q_t'$ where (q_j') is N_j-primary and the N_j's are distinct primes. Then $Q_{iP_i} = I_{P_i} = (q_1')_{P_i} \cdots (q_t')_{P_i}$ so we must have $Q_{iP_i} = (q_j')_{P_i}$ for some j. But Q_i and (q_j') are primary ideals, so $Q_i = (q_j')$. Thus each Q_i is principal. By Theorem 3, $I \subseteq Q_1 \cap \cdots \cap Q_n = Q_1 \cdots Q_n$. Since $Q_1 \cdots Q_n$ is principal, $I = BQ_1 \cdots Q_n$ for some ideal B. Note that B is invertible. Suppose that $B \neq R$. Let P be a minimal prime ideal of B. Since $B \supseteq I$, $P \supseteq$ some P_i. But since B is invertible, replacing I by B in the first few sentences of the proof shows that P has a regular principal P-primary ideal. So by Theorem 4, $P = P_i$. But then $I_{P_i} = B_{P_i}(Q_1 \cdots Q_n)_{P_i} = B_{P_i} Q_{iP_i} = B_{P_i} I_{P_i} \subsetneq P_{iP_i} I_{P_i}$. But this is absurd since I_{P_i} is invertible. Hence $B = R$, so $I = Q_1 \cdots Q_n$ is principal. □

Corollary 11. *Suppose that R is an integral domain or is Noetherian. If every regular nonunit of R is a product of primary elements, then* $\operatorname{Pic}(R) = 0$. □

Proof. If R is either an integral domain or Noetherian, then every invertible ideal is generated by regular elements. Hence every invertible ideal of R is principal. Thus $\operatorname{Pic}(R) = 0$ since for R an integral domain or Noetherian, $\operatorname{Pic}(R)$ is naturally isomorphic to the group of invertible ideals modulo the subgroup of regular principal ideals. □

Remark. The hypothesis in Theorem 10 that I be generated by regular elements is necessary, as the following example from [4] shows. Let D be a Dedekind domain with maximal ideal M such that M is *not* principal but $M^2 = (t)$ is principal. Let $N = \bigoplus \{D/Q \mid Q \neq M$ is a maximal ideal of $D\}$ and let $R = D \oplus N$ be the idealization of D and N. By construction, $\{t^n R\}_{n=0}^{\infty}$ is the set of regular principal ideals of R. Let $P = M \oplus N$. Now $P^2 = tR$, so P is invertible but *not* principal. Since P is maximal, each regular principal ideal $t^n R$ is actually P-primary. Hence each proper regular principal ideal of R is a product of principal primary ideals, but P is an invertible ideal that is not principal.

Theorem 10 can be viewed as a generalization of the well-known result that if R is factorial, then $\operatorname{Pic}(R) = 0$. If R is factorial, then we have the stronger result that every divisorial ideal is principal (i.e., $\operatorname{Cl}(R) = 0$). This result does *not* generalize to weakly factorial domains. For let $R = K[[X, Y]]/(X^2 + Y^3)$ where K is a field. Then R is a one-dimensional local Gorenstein domain that is not a DVR. Thus R is weakly factorial, but $(X, Y)/(X^2 + Y^3)$ is divisorial (in fact every nonzero ideal is divisorial) but is not principal.

For the remainder of the paper R will be an integral domain. A nonzero nonunit element $x \in R$ is *irreducible* if $x = bc$ implies b or c is a unit. Clearly, an irreducible element of a weakly factorial domain must be primary. If every element of R is a product of irreducible elements (e.g., if R has ACC on principal ideals), then R is weakly factorial if and only if every irreducible element is primary. However, every element of a weakly factorial domain need not be a product of irreducible elements. For example, a nondiscrete rank one valuation domain is certainly weakly factorial, but has no irreducible elements.

Theorem 12. *For a one-dimensional Noetherian domain, the following statements are equivalent*:

(1) *R is weakly factorial*;

(2) *Every irreducible element is primary*;

(3) $\mathrm{Pic}(R) = 0$.

If R is a one-dimensional weakly factorial Noetherian domain, then the integral closure $\bar{R}$ of R is a PID.

Proof. The equivalence of (1) and (2) follows from the remarks made in the previous paragraph. Corollary 11 gives the implication $(1) \Rightarrow (3)$. $(3) \Rightarrow (1)$: Let $r \in R$ be a nonunit. If $r = 0$, the result is clear. Suppose $r \neq 0$. Let $(r) = Q_1 \cap \cdots \cap Q_n$ be a reduced primary decomposition for (r). Since $\dim R = 1$, the Q_i's are comaximal, so $(r) = Q_1 \cdots Q_n$. Thus each Q_i is invertible and hence principal since $\mathrm{Pic}(R) = 0$. If $Q_i = (q_i)$, then $r = uq_1 \cdots q_n$ where u is a unit and each q_i is primary.

Suppose that R is a one-dimensional weakly factorial Noetherian domain. Clearly $\bar{R}$ is a Dedekind domain. Hence it suffices to show that $\mathrm{Pic}(\bar{R}) = 0$. To show this it suffices to show that $\mathrm{Pic}(T) = 0$ for each overring $T = R[a_1, \ldots, a_n] \subseteq \bar{R}$. For such a T, the conductor I of T/R is nonzero. Applying the Mayer–Vietoris sequence to the Cartesian square

$$\begin{array}{ccc} R & \hookrightarrow & T \\ \downarrow & & \downarrow \\ R/I & \hookrightarrow & T/I \end{array}$$

we get the exact sequence $\to \mathrm{Pic}(R) \to \mathrm{Pic}(T) \oplus \mathrm{Pic}(R/I) \to \mathrm{Pic}(T/I)$. Now $\mathrm{Pic}(R) = 0$ since R is weakly factorial and $\mathrm{Pic}(T/I) = 0$ since T/I is a zero-dimensional Noetherian ring. By exactness, $\mathrm{Pic}(T) = 0$. (This proof using the Mayer–Vietoris sequence was shown to us by D.F. Anderson.) $\square$

Remark. The above proof actually shows that if R is a one-dimensional Noetherian domain that is weakly factorial and T is any overring of R with $T \subseteq \bar{R}$, then T is again weakly factorial since by the Krull–Akizuki theorem any such T is a one-dimensional Noetherian domain.

Let $R = K[X^2, X^3]$ where K is a field. Then $\bar{R} = K[X]$ is a PID and hence weakly factorial. However, R is not weakly fact·rial since $\operatorname{Pic}(R) \neq 0$. (The ideal $I = (X^2 + X^3, X^4)$ is invertible with inverse $I^{-1} = X^{-2}(1 - X, X^2)$, but is not principal.)

An easy modification of the proof of Theorem 12 yields the following result. For a one-dimensional domain R the following are equivalent: (1) R is weakly factorial, (2) R is Laskerian and $\operatorname{Pic}(R) = 0$, and (3) each non-zero element of R is contained in only finitely many maximal ideals and $\operatorname{Pic}(R) = 0$.

Clearly a one-dimensional quasilocal domain is weakly factorial. Analogous with the fact that a factorial domain is a locally finite intersection of essential DVR's, we show that if R is weakly factorial, then $R = \bigcap_{\operatorname{ht}(P)=1} R_P$ where the intersection is locally finite. This will follow from the next theorem.

Theorem 13. *For an integral domain R, the following statements are equivalent*:

(1) *Every proper principal ideal of R has a primary decomposition involving only height one prime ideals*;

(2) $R = \bigcap_{\operatorname{ht}(P)=1} R_P$ *where the intersection is locally finite.*

Proof. (2) $\Rightarrow$ (1). Let $0 \neq x \in R$ be a nonunit. Since the intersection is locally finite, x is contained in only finitely many height one prime ideals $P_1, \ldots, P_n$. Let $Q_i = R_{P_i} x \cap R$, so Q_i is P_i-primary. Then

$$\begin{aligned} Rx &= \Big(\bigcap_{\operatorname{ht}(P)=1} R_P\Big)x = \bigcap_{\operatorname{ht}(P)=1} R_P x = \Big(\bigcap_{\operatorname{ht}(P)=1} R_P x\Big) \cap R \\ &= \bigcap_{\operatorname{ht}(P)=1} (R_P x \cap R) = Q_1 \cap \cdots \cap Q_n . \end{aligned}$$

(1) $\Rightarrow$ (2). Let $\mathscr{P}$ be the set of height one prime ideals of R. Then $R = \bigcap_{P \in \mathscr{P}} R_P$ if and only if each proper ideal of the form $(a):(b)$ is contained in some $P \in \mathscr{P}$ [5, Exercise 22, p. 52]. Let $(a) = Q_1 \cap \cdots \cap Q_n$ where Q_i is P_i-primary and $\operatorname{ht}(P_i) = 1$. Then $(a):(b) = (Q_1 : (b)) \cap \cdots \cap (Q_n : (b))$ and each $Q_i : (b)$ is either P_i-primary or is R. Hence $(a):(b) \subset$ some P_i. The intersection is locally finite since each nonzero nonunit of R is contained in only finitely many height one primes, namely those occurring in the primary decomposition. □

Corollary 14. *Suppose that R is weakly factorial. Then $R = \bigcap_{\operatorname{ht}(P)=1} R_P$ is locally finite.* □

Theorem 15. *Suppose that R is weakly factorial. Then the following statements are equivalent*:

(1) *R is factorial*;
(2) *R is a Krull domain*;
(3) *For each height one prime ideal P, R_P is a DVR.*

Proof. It is well known that (1) ⇒ (2) ⇒ (3). Corollary 14 gives (3) ⇒ (2). (2) ⇒ (1). Let P be a height one prime ideal of R. Let $x \in P - P^2$. Now (x) is a product of principal primary ideals. Since $x \in P - P^2$, the primary component for P must be P. Thus P is principal. Since every height one prime ideal of R is principal, R is factorial. □

Corollary 16. *Let R be a Noetherian weakly factorial domain. Then R is factorial if and only if R is integrally closed.*

Proof. A Noetherian domain is integrally closed if and only if it is a Krull domain. □

It is well known that an integral domain R is factorial if and only if $R[X]$ is factorial. It is easily seen that $R[X]$ weakly factorial implies that R is weakly factorial. However, R can be weakly factorial without $R[X]$ being weakly factorial. For example, let $R = K[[T^2, T^3]]$ where K is a field. Then R is a one-dimensional local ring and hence is weakly factorial. However, since $\mathrm{Pic}(R[X]) \neq 0$ (for R is not seminormal), $R[X]$ is not weakly factorial. It is interesting to note that $\overline{R[X]} = \bar{R}[X] = K[[T]][X]$ is factorial. The exact condition necessary for $R[X]$ to be weakly factorial is given in the next theorem.

Theorem 17. *For an integral domain R the following statements are equivalent*:
(1) *$R[X]$ is weakly factorial*;
(2) *R is a weakly factorial GCD domain.*

Proof. (1) ⇒ (2). Clearly $R[X]$ weakly factorial implies that R is weakly factorial. Suppose that $aX + b$ is irreducible in $R[X]$. Then $(aX + b)$ is primary. But then in $K[X] = R[X]_S$ where $S = R - \{0\}$, $(aX + b)K[X]$ is primary and irreducible and hence prime. Thus $(aX + b)$ is a prime ideal of $R[X]$. But $(aX + b)$ prime implies that $(a) \cap (b) = (ab)$ [5, Exercise 9, p. 99]. Now let $a, b \in R - \{0\}$ be arbitrary, we can write $aX + b = r(a'X + b')$ where r is a product of primaries from R and $a'X + b'$ is primary. Then $a'X + b'$ is prime so $(a') \cap (b') = (a'b')$. Hence $(a) \cap (b) = (ra') \cap (rb') = (r)((a') \cap (b')) = (r)(a'b')$ is principal. Since the intersection of any two principal ideals of R is principal, R is a GCD domain.

(2) ⇒ (1). Suppose that R is a weakly factorial GCD domain. Let $0 \neq f \in R[X]$. Since R is a GCD domain, we can write $f = rf'$ where $(A_{f'})_v = R$ [5, Theorem 34.10]. Since R is weakly factorial, $r \in R$ is either a unit or is a product of primary elements. Now f' can be written as a product of irreducible polynomials. But $R[X]$ is a GCD domain and in a GCD domain every irreducible element is actually prime [5, Pro-

position 46.2], so $f = rf'$ is a product of primary elements. Hence $R[X]$ is weakly factorial. □

The next theorem provides another characterization of weakly factorial GCD domains. Observe that Theorem 18 implies that a weakly factorial GCD domain is completely integrally closed.

Theorem 18. *Suppose that R is weakly factorial. Then R is a GCD domain if and only if for each height one prime ideal P of R, R_P is a valuation domain.*

Proof. '$\Rightarrow$'. This implication does not require that R be weakly factorial. Suppose that R is a GCD domain and P is a height one prime ideal of R. Then R_P is a one-dimensional GCD domain and hence is a valuation domain [6].

'$\Leftarrow$'. Assume that R is weakly factorial and that for each height one prime P, R_P is a valuation domain. To show that R is a GCD domain, it suffices to show that for nonzero nonunits $a, b \in R$, $(a) \cap (b)$ is principal. Writing a and b as products of primaries with distinct radicals, we easily see that it suffices to show that $(q) \cap (q')$ is principal where (q) and (q') are both P-primary. Here P is necessarily a height one prime. But in the valuation domain R_P, $(q)_P$ and $(q')_P$ are comparable, say $(q)_P \subseteq (q')_P$. But then $(q) \subseteq (q')$ since (q) and (q') are primary. □

Corollary 19. *Suppose that $R[X]$ is weakly factorial. Then the following statements are equivalent*:

(1) *$R[X]$ is factorial*;

(2) *$R[X]$ has ACC on principal ideals*;

(3) *Every height one prime ideal of $R[X]$ is finitely generated.*

Proof. Clearly (1) $\Rightarrow$ (2),(3). (2) $\Rightarrow$ (1). If $R[X]$ is weakly factorial, then by Theorem 17, R is a GCD domain and hence $R[X]$ is a GCD domain. But a GCD domain with ACC on principal ideals is factorial. (3) $\Rightarrow$ (1). Let P be a height one prime ideal of $R[X]$. Since P is finitely generated and $R[X]$ is a GCD domain, $R[X]_P$ is a DVR. By Theorem 15, $R[X]$ is factorial. □

Remark. In Corollary 19, it is easily seen that in statements (1), (2) and (3), $R[X]$ can be replaced by R. For R is factorial if and only if $R[X]$ is factorial and R has ACC on principal ideals if and only if $R[X]$ does. If in (3), we assume that every height one prime ideal of R is finitely generated, then the same proof as (3) $\Rightarrow$ (1) shows that R is factorial.

An integral domain R is called a *generalized Krull domain* if $R = \bigcap_\lambda V_\lambda$ where each V_λ is an essential rank one valuation overring of R and the intersection is locally finite. Then $\{V_\lambda\} = \{R_P \mid \mathrm{ht}(P) = 1\}$ by [5, Proposition 43.7]. Theorem 18 and Corollary 14 show that a weakly factorial GCD domain is a generalized Krull domain. Our next result gives the converse.

Theorem 20. *For an integral domain R the following statements are equivalent*:
(1) *R is a weakly factorial GCD domain*;
(2) *R is a weakly factorial generalized Krull domain*;
(3) *R is a generalized Krull domain and a GCD domain.*

Proof. (1) $\Rightarrow$ (2). Theorem 18 and Corollary 14. (2) $\Rightarrow$ (1). Theorem 18. (2) $\Rightarrow$ (3). Theorem 18. (3) $\Rightarrow$ (1). Now $R = \bigcap_{\mathrm{ht}(P)=1} R_P$ (locally finite) and each R_P is a valuation domain. Let v_P be the valuation corresponding to R_P. Let $0 \neq x \in R$ be a nonunit. Let $P_1, \ldots, P_n$ be the minimal prime ideals containing x. Observe that since $(x) = Q_1 \cap \cdots \cap Q_n$ where $Q_i = (x)_{P_i} \cap R$ and Q_i is P_i-primary, if $n = 1$, then (x) is P_1-primary. We show by induction on n that x is a product of primary elements. We have just done the case $n = 1$. Now by the Approximation Theorem [5, Exercise 1, p. 553], there exists an element y in the quotient field of R with $v_{P_1}(x) = v_{P_1}(y)$, $v_{P_i}(y) = 0$ for $i = 2, \ldots, n$ and $v_Q(y) \geq 0$ for all other height one primes Q of R. Then $y \in R$. Let x_1 be the GCD of x and y and let $x = x_1 x_2$. Note that $v_{P_1}(x) = v_{P_1}(x_1)$ and $v_{P_1}(x_2) = 0$. Also $v_{P_i}(x_1) = 0$ for $i = 2, \ldots, n$. Thus P_1 is the only minimal prime ideal containing x_1, so (x_1) is P_1-primary. Now x_2 is contained in only $n-1$ minimal primes, namely $P_2, \ldots, P_n$, so by induction x_2 is also a product of primary elements. Hence $x = x_1 x_2$ is a product of primary elements. $\square$

Our last result relates the weakly factorial property to the group of divisibility of an integral domain. Let R be an integral domain with quotient field K. Let U_R be the group of units of R and $K^* = K - \{0\}$. The *group of divisibility of R* is the quotient group $G(R) = K^*/U_R$ which is partially ordered by the relation $xU_R \leq yU_R$ if and only if $y/x \in R$. Theorem 21 generalizes the well-known result that R is factorial if and only if $G(R)$ is order isomorphic to a direct sum of copies of $\mathbb{Z}$ where $\mathbb{Z}$ has the usual order and the direct sum is given the usual order.

Theorem 21. *Suppose that $G(R)$ is order isomorphic to $\bigoplus_\lambda H_\lambda$ where each H_λ is a rank one totally ordered abelian group. Then R is a weakly factorial GCD domain. If $R = \bigcap_{\mathrm{ht}(P)=1} R_{P_\lambda}$ is a weakly factorial GCD domain, then $G(R)$ is order isomorphic to $\bigoplus_\lambda G(R_{P_\lambda})$ where each $G(R_{P_\lambda})$ is a rank one totally ordered abelian group. Moreover, given any collection $\{H_\lambda\}$ of rank one totally ordered abelian groups, there is a weakly factorial Bézout domain R with $G(R)$ order isomorphic to $\bigoplus H_\lambda$.*

Proof. Suppose that $\psi : G(R) \to \bigoplus_\lambda H_\lambda$ is an order preserving isomorphism. Let $\psi_\lambda = p_\lambda \psi$ where $p_\lambda : \bigoplus H_\lambda \to H_\lambda$ is the λ-projection map. First, since $G(R)$ is lattice-ordered, R is a GCD domain. Let $P_\lambda = \{r \in R \mid \psi_\lambda(r) > 0\}$, so P_λ is a prime ideal of R. Suppose that $0 \neq r \in R$ with $\psi_{\lambda_0}(r) > 0$ and $\psi_{\lambda_\alpha}(r) = 0$ for $\lambda_\alpha \neq \lambda_0$. We claim that (r) is P_{λ_0}-primary. Let $x \in P_{\lambda_0}$, so $\psi_{\lambda_0}(x) > 0$. Since H_λ has rank one, it is Archimedian, so there is a positive integer n with $n\psi_{\lambda_0}(x) > \psi_{\lambda_0}(r)$. Hence

$n\psi(x)>\psi(r)$ or $\psi(x^n)>\psi(r)$ so $(x^n)\subseteq(r)$. Thus $\sqrt{(r)}=P_{\lambda_0}$. Suppose that $ab\in(r)$, say $ab=rt$ where $b\notin P_{\lambda_0}$. Then $\psi_{\lambda_0}(b)=0$ and $\psi_{\lambda_0}(a)=\psi_{\lambda_0}(a)+\psi_{\lambda_0}(b)=\psi_{\lambda_0}(ab)=\psi_{\lambda_0}(rt)=\psi_{\lambda_0}(r)+\psi_{\lambda_0}(t)\geq\psi_{\lambda_0}(r)$. For all other λ_α, $\psi_{\lambda_\alpha}(a)\geq 0=\psi_{\lambda_\alpha}(r)$, so $\psi(a)\geq\psi(r)$ and hence $a\in(a)\subseteq(r)$; thus (r) is P_{λ_0}-primary. Now let $0\neq x\in R$ be a nonunit. Then $\psi(x)=h_{\alpha_1}+\cdots+h_{\alpha_n}$ where $h_{\alpha_i}\in H^+_{\alpha_i}-\{0\}$. Let $r_i\in R$ with $\psi(r_i)=h_{\alpha_i}$. Then by the preceding remarks, (r_i) is P_{α_i}-primary. Hence $r=ur_1\cdots r_n$ where u is a unit and each r_i is primary. Thus R is weakly factorial.

Suppose that $R=\bigcap_{\mathrm{ht}(P)=1}R_{P_\lambda}=1$ is a weakly factorial GCD domain. Let $H_\lambda=G(R_{P_\lambda})$. Since R_{P_λ} is a one-dimensional valuation domain, $G(R_{P_\lambda})$ is a rank one totally ordered abelian group. Define the map $\psi:G(R)\to\bigoplus H_\lambda$ by $aU\to(aU_{R_{P_\lambda}})$. Since the intersection is locally finite, the image of ψ actually lies in $\bigoplus H_\lambda$. Clearly ψ is an order preserving monomorphism. To show that ψ is surjective it suffices to show that for $0<h_\lambda\in H_\lambda$, $h_\lambda\in\operatorname{im}\psi$. Let $h_\lambda=rU_{R_{P_\lambda}}$ where $r\in R$. Now $(r)_{P_\lambda}$ is P_λ-primary. Hence $(r)_{P_\lambda}\cap R$ is a principal primary ideal, say (q_λ). Then $\psi(q_\lambda)=h_\lambda$.

Finally, suppose that $\{H_\lambda\}$ is a collection of rank one totally ordered abelian groups. Then $H=\bigoplus H_\lambda$ is a lattice ordered abelian group. By the Krull-Kaplansky-Jaffard-Ohm Theorem [5, Proposition 18.6] there is a Bézout domain R with $G(R)\simeq\bigoplus H_\lambda$. By the first result of the theorem, R is a weakly factorial Bézout domain. $\square$

Remark. A generalized Krull domain R that is also Bézout necessarily has Krull dimension one, since each maximal ideal of R contains a unique height one prime ideal. Hence the examples of weakly factorial GCD domains constructed by the Krull-Kaplansky-Jaffard-Ohm Theorem are all one-dimensional. Higher-dimensional examples may be obtained by the adjunction of indeterminates since by Theorem 17, $R[X]$ is a weakly factorial GCD domain if R is.

We end this paper with some examples. Since a Noetherian weakly factorial domain is factorial if and only if it is integrally closed and since a weakly factorial GCD domain is completely integrally closed, it is reasonable to conjecture that a weakly factorial domain is a GCD domain if it is completely integrally closed. This is however not the case. It is well known that there exist completely integrally closed one-dimensional quasilocal domains (which are necessarily weakly factorial) that are not valuation domains (and hence not GCD domains). Also, a completely integrally closed GCD domain need not be weakly factorial. The ring of algebraic integers is a one-dimensional (and hence completely integrally closed) Bézout domain but it is not weakly factorial since each nonunit belongs to infinitely many maximal ideals [5, Proposition 42.8]. A second example of a completely integrally closed Bézout domain that is not weakly factorial is the ring E of entire functions. It is interesting to note that $\{z-\alpha\mid\alpha\in\mathbb{C}\}$ is the complete set of nonassociative prime elements of E, that $\{(z-\alpha)\mid\alpha\in\mathbb{C}\}$ is the set of nonzero proper finitely generated prime ideals of E and that $\bigcup_{\alpha\in\mathbb{C}}(z-\alpha)$ is the set of nonunits of E [5, pp. 147–148].

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